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On the Superposition of Electromagnetic Waves in General Relativity

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Abstract

It is shown that, in a region of space-time containing two independent electromagnetic waves propagating in different directions, it is not possible for the two waves to follow simultaneously affinely parameterised shear-free and twist-free null geodesic congruences.

1. *Introduction*

In flat space-time the linearity of Maxwell's equations implies that solutions may be simply superposed. That is, there is no interaction between electromagnetic waves. In general relativity however, the coupled Einstein–Maxwell equations are highly non-linear and a simple superposition of solutions will in general no longer be possible. That is, in general relativity there will be a non-linear gravitational interaction between electromagnetic waves.

It is obviously of considerable importance to investigate the nature of this non-linear interaction. Such an investigation will lead to a greater understanding of the general theory of relativity. But of more direct importance are its applications to cosmology. Almost all astronomical observations are made in terms of electromagnetic waves. It is generally assumed that light received even from a near star will have travelled millions of light years before being observed. During this time it will have interacted with (or passed through) a considerable number of similar electromagnetic waves. If a cumulative non-linear interaction occurs then all observations will be seriously affected. Unfortunately any such non-linear gravitational interaction will be undetectable in laboratory experiments. It is therefore important to investigate this property theoretically and to search for both approximate and exact solutions.

An approximate approach to this subject has been discussed by Penrose (1966). He has suggested that when a null ray passes through a region of non-zero energy density it will tend to be focused: the focusing power being proportioned to the energy density. This may be applied to energies due to the presence of matter, electromagnetism or gravitation. Some cosmological applications of this approach have been mentioned by Bertotti (1966). The implications to the present topic are that when two electromagnetic waves pass through each other they will tend to focus each other. It may therefore be expected that foci will occur. These will appear as singularities in the space-time.

The present work is motivated by a desire to obtain an exact solution of the field equations describing two interacting null electromagnetic waves. A simple expansion-free solution is already known (Griffiths, 1974). It is shown here that an equivalent simple twist-free solution is not possible.

2. *Geometrical Considerations*

In this section I consider the possibility of finding an exact solution of the Einstein–Maxwell equations describing two interacting null electromagnetic waves. The most likely present method of finding exact solutions is to define a null tetrad of basis vectors in terms of the principal null congruences of the field and to use its associated spin coefficients. This method will be followed using the formalism of Newman & Penrose (1962). Reference should be made to this paper for definition of notation, and equations quoted directly from it will be prefixed by NP. This method however has only been successful for space-times which contain a principal null congruence with particularly simple geometrical properties. For example its tangent vector may be geodesic, shear-free and affinely parameterised. It is therefore necessary initially to discuss any geometrical simplifications that can be made. I am considering a region of space-time containing two electromagnetic waves. Such a region will possess a finite boundary and therefore asymptotic properties are not important.

Assuming that the electromagnetic waves are propagating in different directions, it is possible to choose the two null basis vectors l_μ and n_μ to be tangent to the two waves. In the notation of Newman and Penrose the components of the electromagnetic field will now satisfy

$$\Phi_0 \neq 0, \qquad \Phi_1 = 0, \qquad \Phi_2 \neq 0$$

The electromagnetic field thus has two distinct principal null vectors l_μ and n_μ, whose directions are now fixed.

It is also reasonable to assume that both waves satisfy Maxwell's equations independently. That is, there is no electromagnetic interaction between the two waves. Maxwell's equations thus take the form

$$D\Phi_2 = (\rho - 2\epsilon)\Phi_2$$

$$\delta\Phi_2 = (\tau - 2\beta)\Phi_2, \qquad \kappa = \sigma = 0$$

$$\Delta\Phi_0 = -\ (\mu - 2\gamma)\Phi_0$$

$$\bar{\delta}\Phi_0 = -(\pi - 2\alpha)\Phi_0, \qquad \nu = \lambda = 0$$

It can thus be seen that both waves continue to follow shear-free null geodesics as they do in vacuum (Mariot, 1954; Robinson, 1961).

It is also convenient to assume that there exist affine parameters defined simultaneously along the two congruences. This is equivalent to assuming that

$$\epsilon + \bar{\epsilon} = 0, \qquad \gamma + \bar{\gamma} = 0$$

The remaining tetrad freedom is

$$l'_\mu = R l_\mu, \qquad m'_\mu = e^{iS} m_\mu, \qquad n'_\mu = R^{-1} n_\mu$$

subject to the restrictions

$$DR = 0, \qquad \Delta R = 0$$

It can be seen from the commutation relations (NP 4.4) that these equations are only integrable provided

$$\tau + \bar{\pi} = 0$$

Under this condition it is also possible to choose S such that

$$\epsilon - \bar{\epsilon} = 0, \qquad \gamma - \bar{\gamma} = 0$$

I will also assume that the congruences are twist-free. That is,

$$\rho = \bar{\rho}, \qquad \mu = \bar{\mu}$$

Under this assumption both congruences are hypersurface orthogonal.

3. *The Main Theorem*

It will now be shown that the four assumptions of the previous section are inconsistent. That is, there exist no vacuum solutions of the Einstein-Maxwell equations under the assumptions

$$\Phi_1 = 0 \tag{3.1}$$

$$\kappa = \sigma = \nu = \lambda = 0 \tag{3.2}$$

$$\epsilon = \gamma = \tau + \bar{\pi} = 0 \tag{3.3}$$

$$\rho = \bar{\rho}, \qquad \mu = \bar{\mu} \tag{3.4}$$

Therefore, when two electromagnetic waves pass through each other, they will not follow affinely parameterised, shear-free and twist-free null geodesics.

Substituting the above assumptions into the field equations (NP 4.2) gives

$$\Psi_0 = 0, \qquad \Psi_2 = \tau\bar{\tau}, \qquad \Psi_4 = 0$$

$$D\rho = \rho^2 + \Phi_{00}, \qquad \Delta\mu = -\mu^2 - \Phi_{22}$$

$$D\tau = \Psi_1, \qquad \Delta\tau = \bar{\Psi}_3$$

$$D\alpha = \rho(\alpha - \bar{\tau}), \qquad \Delta\alpha = -\mu\alpha - \Psi_3$$

$$D\beta = \rho\beta + \Psi_1, \qquad \Delta\beta = -\mu(\beta + \tau)$$

$$D\mu + \delta\bar{\tau} = \rho\mu + 2\tau\bar{\tau} + \bar{\tau}(\bar{\alpha} - \beta)$$

$$D\mu + \Delta\rho = 0$$

$$\delta\rho = \rho(\bar{\alpha}+\beta) - \Psi_1, \qquad \delta\mu = -\mu(\bar{\alpha}+\beta) + \Psi_3$$
$$\delta\tau = \tau(\tau+\beta-\bar{\alpha}) + \Phi_{02}$$
$$\delta\alpha - \bar{\delta}\beta = \rho\mu + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta - \tau\bar{\tau}$$

Using these and Maxwell's equations the Bianchi identities (NP A3) become

$$D\Psi_1 + \bar{\Phi}_0\delta\Phi_0 = 4\rho\Psi_1 + 2\beta\Phi_{00} \tag{3.5}$$
$$\delta\Psi_1 - \bar{\Phi}_2 D\Phi_0 = (4\tau + 2\beta)\Psi_1 \tag{3.6}$$
$$\bar{\delta}\Psi_1 = (3\bar{\tau} + 2\alpha)\Psi_1 + \tau\bar{\Psi}_1 - \mu\Phi_{00} - 3\rho\tau\bar{\tau} \tag{3.7}$$
$$\Delta\Psi_1 + \tau D\mu = -2\mu\Psi_1 + \rho\mu\tau \tag{3.8}$$
$$D\Psi_3 + \bar{\tau}D\mu = 2\rho\Psi_3 + \rho\mu\bar{\tau} \tag{3.9}$$
$$\delta\Psi_3 = (3\tau - 2\beta)\Psi_3 + \bar{\tau}\bar{\Psi}_3 + \rho\Phi_{22} + 3\mu\tau\bar{\tau} \tag{3.10}$$
$$\bar{\delta}\Psi_3 - \bar{\Phi}_0\Delta\Phi_2 = (4\bar{\tau} - 2\alpha)\Psi_3 \tag{3.11}$$
$$\Delta\Psi_3 + \bar{\Phi}_2\bar{\delta}\Phi_2 = -4\mu\Psi_3 - 2\alpha\Phi_{22} \tag{3.12}$$

The commutation relations (NP 4.4) are

$$(\Delta D - D\Delta)\phi = 0$$
$$(\delta D - D\delta)\phi = (\bar{\alpha} + \beta + \tau)D\phi - \rho\delta\phi$$
$$(\delta\Delta - \Delta\delta)\phi = (\tau - \bar{\alpha} - \beta)\Delta\phi + \mu\delta\phi$$
$$(\bar{\delta}\delta - \delta\bar{\delta})\phi = (\beta - \bar{\alpha})\bar{\delta}\phi + (\alpha - \bar{\beta})\delta\phi$$

Applying these to the Bianchi identities (3.5) and (3.8) (or (3.6) and (3.7)), (3.9) and (3.12) (or (3.10) and (3.11)), (3.7) and (3.8) and (3.9) and (3.10) gives

$$\mu\bar{\Phi}_0\delta\Phi_0 = (D\mu + \rho\mu)\Psi_1 - \Phi_{00}\bar{\Psi}_3 + 2\mu(\tau+\beta)\Phi_{00} + 3\rho\tau(D\mu - \rho\mu)$$
$$\rho\bar{\Phi}_2\bar{\delta}\Phi_2 = -(D\mu + \rho\mu)\Psi_3 + \Phi_{22}\bar{\Psi}_1 + 2\rho(\bar{\tau} - \alpha)\Phi_{22} + 3\mu\tau(D\mu - \rho\mu)$$
$$(D\mu - \rho\mu)^2 + 4\rho\mu\tau\bar{\tau} - 3\Psi_1\Psi_3 - \bar{\Psi}_1\bar{\Psi}_3 - \Phi_{00}\Phi_{22} - \mu\bar{\tau}\Psi_1 - \mu\tau\bar{\Psi}_1 + 3\rho\tau\Psi_3 + 3\rho\bar{\tau}\bar{\Psi}_3 = 0 \tag{3.13}$$
$$(D\mu - \rho\mu)^2 + 4\rho\mu\tau\bar{\tau} - 3\Psi_1\Psi_3 - \bar{\Psi}_1\bar{\Psi}_3 - \Phi_{00}\Phi_{22} - 3\mu\bar{\tau}\Psi_1 - 3\mu\tau\bar{\Psi}_1 + \rho\tau\Psi_3 + \rho\bar{\tau}\bar{\Psi}_3 = 0$$

The last two equations imply that

$$\Psi_1\Psi_3 = \bar{\Psi}_1\bar{\Psi}_3$$

and

$$\mu(\bar{\tau}\Psi_1 + \tau\bar{\Psi}_1) + \rho(\tau\Psi_3 + \bar{\tau}\bar{\Psi}_3) = 0$$

From these it is seen that either

(a) $$\mu\bar{\Psi}_1 + \rho\Psi_3 = 0$$

or

(b) $$\bar{\tau}\Psi_1 + \tau\bar{\Psi}_1 = 0 = \tau\Psi_3 + \bar{\tau}\bar{\Psi}_3$$

These are now considered one at a time.

Case (a)

$$\mu\bar{\Psi}_1 + \rho\Psi_3 = 0 \tag{3.14}$$

Operating on this with δ and D gives

$$\mu^2\Phi_{00} = \rho^2\Phi_{22} \tag{3.15}$$

and

$$2\rho\bar{\tau}(D\mu - \rho\mu) = \Phi_{00}(\Psi_3 - \mu\bar{\tau}) \tag{3.16}$$

which implies that

$$\tau\bar{\Psi}_1 = \bar{\tau}\Psi_1, \qquad \tau\Psi_3 = \overline{\tau\Psi}_3 \tag{3.17}$$

Equations (3.13), (3.14), (3.16) and (3.17) give the condition

$$(\Psi_1 - \rho\tau)\{\mu\Phi_{00}^2(\Psi_1 + 3\rho\tau) + 16\rho^3\tau\bar{\tau}(\Psi_1 - \rho\tau)\} = 0 \tag{3.18}$$

Equation (3.15) implies that

$$D\Phi_{00} = \Phi_{00}\left\{\frac{\Phi_{00}\Psi_1}{\rho^2\tau} + 2\rho + \frac{3\Phi_{00}}{\rho}\right\}$$

and using either of the factors of (3.18) it may be shown, after a short calculation, that this is not consistent with the condition

$$\Delta\Phi_{00} = -2\mu\Phi_{00} \tag{3.19}$$

which is required by Maxwell's equations. Thus there is no solution under case (a).

Case (b)

$$\bar{\tau}\Psi_1 + \tau\bar{\Psi}_1 = 0 = \tau\Psi_3 + \bar{\tau}\bar{\Psi}_3$$

Proceeding in a similar manner, operating on this with D and Δ gives

$$D\mu = \rho\mu + \frac{\Psi_1\Psi_3}{\tau\bar{\tau}}$$

$$\Psi_1\bar{\Psi}_1 - 2\tau\bar{\tau}\Phi_{00} - 3\frac{\rho}{\mu}\Psi_1\Psi_3 = 0$$

$$\Psi_3\bar{\Psi}_3 - 2\tau\bar{\tau}\Phi_{22} - 3\frac{\mu}{\rho}\Psi_1\Psi_3 = 0$$

Operating on the last of these twice with D, using these identities gives

$$4\rho^2 \tau\bar{\tau}\Phi_{22} = 3\Phi_{00}\left(\Psi_3\bar{\Psi}_3 - \frac{\mu}{\rho}\Psi_1\Psi_3\right)$$

and

$$D\Phi_{00} = \frac{3}{\rho}\Phi_{00}^2$$

which again can be shown to be inconsistent with (3.19) after a short calculation.

It may thus be concluded that no exact solutions of the Einstein-Maxwell equations exist which satisfy the conditions (3.1), (3.2), (3.3) and (3.4).

4. *Discussion*

It has been shown that the four geometrical conditions mentioned in Section 2 are not mutually consistent, and therefore in an exact solution at least one must be broken. Conditions (3.1) and (3.2) are just the conditions that a space-time admits two independent electromagnetic waves, the null vectors l_μ and n_μ having been aligned with them. These conditions must be retained in this case. Conditions (3.3) and (3.4) cannot therefore both be satisfied. Thus either the two congruences will not be simultaneously affinely parameterised, or they will have non-zero twist, or both.

The most simple case will be that in which both congruences are affinely parameterised and have non-zero twist. In this case the affine parameters u and v may be chosen as coordinates x^1 and x^2 respectively so that

$$l^\mu = \delta_2{}^\mu, \qquad n^\mu = \delta_1{}^\mu$$

$$D = \frac{\partial}{\partial v}, \qquad \Delta = \frac{\partial}{\partial u}$$

An exact solution corresponding to this case but with zero expansion has been given by Griffiths (1974).

The alternative case with twist-free congruences may similarly be simplified. In this case the propagation vectors are scalar multiples of a gradient and it is convenient to choose coordinates such that

$$l_\mu = A\delta_\mu{}^1, \qquad n_\mu = B\delta_\mu{}^2$$

where A and B are scalar functions of the coordinates.

In either case we may write $\rho = \theta + i\omega$ where θ is the convergence of one congruence and ω its twist (or rotation), and the field equation (NP 4.2a) becomes

$$D\theta = \theta^2 - \omega^2 + \Phi_{00}$$

$$D\omega = 2\theta\omega$$

Thus in the first case when ω is non-zero, it takes the form

$$\omega = \omega^0 \exp(2\int \theta\, dv)$$

where ω^0 is independent of v. It can be seen from these equations that the focusing effect of Φ_{00} is quickly countered by the effect of the term $-\omega^2$ and the kind of simple focusing described by Penrose will not occur. In the alternative case when ω is zero, it will still be possible to choose an affine parameter along one congruence and a mutual focusing effect may be inferred.

Obviously more work needs to be done on this subject and it is hoped that further exact solutions will soon be obtained. It would be interesting to see if results similar to those given here apply to gravitational waves. Namely, when two gravitational waves pass through each other, can they follow simultaneously affinely parameterised twist-free null geodesics? For the known exact solutions describing this case, in those given by Szekeres (1972) the waves are not affinely parameterised, and those given by Griffiths (1975) are not twist-free.

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